Set Comprehension in Church's Type Theory

Chad E. Brown

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Simple Types

Simple Types $\mathcal{T}$:

o	(truth values)
ι	(individuals)
$(\alpha\beta)$	(functions from β to α)

$(\alpha\beta\gamma)$ abbreviates $((\alpha\beta)\gamma)$

A Standard Frame:

$$\begin{aligned} \mathcal{D}_o &= \{\mathtt{T}, \mathtt{F}\}. \\ \mathcal{D}_\iota &= \mathbb{N} \text{ (natural numbers).} \\ \mathcal{D}_{\alpha\beta} &= \mathcal{D}_\alpha^{\mathcal{D}_\beta}\text{, all functions from } \mathcal{D}_\beta \text{ to } \mathcal{D}_\alpha. \end{aligned}$$

$\mathcal{D}_{o\iota} \cong \mathcal{P}(\mathbb{N})$: $X \subseteq \mathbb{N} \leftrightarrow \chi_X : \mathbb{N} \to \{\mathtt{T}, \mathtt{F}\}$ (characteristic functions)

$\mathcal{D}_{o\iota\iota} \cong \mathcal{P}(\mathbb{N} \times \mathbb{N})$: Binary relations on $\mathbb{N}$

$\mathcal{D}_{o(o\iota)} \cong \mathcal{P}(\mathcal{P}(\mathbb{N}))$

Simply Typed λ-Calculus

	x_α	Variables ($\mathcal{V}$)
	A_α	Parameters ($\mathcal{P}$)
Terms:	c_α	Logical Constants ($\mathcal{S}$)
	$[\mathbf{F}_{\alpha\beta}\,\mathbf{B}_\beta]_\alpha$	Application
	$[\lambda y_\beta\,\mathbf{A}_\alpha]_{\alpha\beta}$	λ-abstraction

Examples:

$[\lambda y_\beta\, A_\alpha]_{\alpha\beta}$	Constant A function
$[\lambda x_\alpha\, x]_{\alpha\alpha}$	Identity function
$[\lambda u_{o\alpha}\,[u\, A_\alpha]]_{o(o\alpha)}$	Set of sets containing A: $\left\{u_{o\alpha} \,\middle|\, A \in u\right\}$

Dot Convention: A dot stands for a left bracket whose mate is as far to the right as possible without changing the existing bracketing. $[\lambda u_{o\alpha}\mathbf{.}\, u\, A_\alpha]$ means $[\lambda u_{o\alpha}\,[u\, A_\alpha]]$

Conversion and Reduction

Equality of terms: $\alpha\beta\eta$

α-conversion	Changing Bound Variables	
β-reduction	$[[\lambda y_\beta\, \mathbf{A}_\alpha]\, \mathbf{B}] \stackrel{\beta}{\longrightarrow} [\mathbf{B}/y]\mathbf{A}$	
η-reduction	$[\lambda y_\beta\, [\mathbf{F}_{\alpha\beta}\, y]] \stackrel{\eta}{\longrightarrow} \mathbf{F}$	$(y_\beta \notin \mathbf{Free}(\mathbf{F}))$

Every term has a unique $\beta\eta$-normal form,

up to α-conversion.

Logical Constants

Some logical constants which may be in $\mathcal{S}$:

$\top_o$	true
$\bot_o$	false
$\neg_{oo}$	negation
$\vee_{ooo}$	disjunction
$\wedge_{ooo}$	conjunction
$\supset_{ooo}$	implication
$\equiv_{ooo}$	equivalence
$=^{\alpha}_{o\alpha\alpha}$	equality at type α
$\Pi^{\alpha}_{o(o\alpha)}$	universal quantification over type α
$\Sigma^{\alpha}_{o(o\alpha)}$	existential quantification over type α

Intuition: $[\Sigma^{\alpha}.\lambda x_\alpha \mathbf{C}_o]$ maps to true iff $\{x_\alpha \mid \mathbf{C}\}$ is nonempty.

Abbreviations for Logical Operators

$$[\mathbf{A}_o \lor \mathbf{B}_o] \quad \text{means} \quad [\lor_{ooo} \mathbf{A}_o \mathbf{B}_o]$$

$$[\mathbf{A}_o \land \mathbf{B}_o] \quad \text{means} \quad [\land_{ooo} \mathbf{A}_o \mathbf{B}_o]$$

$$[\mathbf{A}_o \supset \mathbf{B}_o] \quad \text{means} \quad [\supset_{ooo} \mathbf{A}_o \mathbf{B}_o]$$

$$[\mathbf{A}_o \equiv \mathbf{B}_o] \quad \text{means} \quad [\equiv_{ooo} \mathbf{A}_o \mathbf{B}_o]$$

$$[\mathbf{A}_\alpha =^\alpha \mathbf{B}_\alpha] \quad \text{means} \quad [=^\alpha_{o\alpha\alpha} \mathbf{A}_\alpha \mathbf{B}_\alpha]$$

$$[\forall x_\alpha \mathbf{A}_o] \quad \text{means} \quad [\Pi^\alpha_{o(o\alpha)} \mathbf{.} \lambda x_\alpha \mathbf{A}_o].$$

$$[\exists x_\alpha \mathbf{A}_o] \quad \text{means} \quad [\Sigma^\alpha_{o(o\alpha)} \mathbf{.} \lambda x_\alpha \mathbf{A}_o].$$

Church's Type Theory

Church's Type Theory:

- Simply typed λ-calculus with the signature $\{\neg, \vee\} \cup \{\Pi^\alpha \mid \alpha \in \mathcal{T}\}$ (and perhaps a description or choice operator).
- Axioms of Extensionality
- Axiom of Description or Choice
- Axiom of Infinity

Extensional Type Theory

$\mathcal{S}$ Fragment of Extensional Type Theory:

- Simply typed λ-calculus with the signature $\mathcal{S}$
- Extensionality
- No Axiom of Description or Choice
- No Axiom of Infinity

Comprehension Principles

The type $(o\alpha)$ is the type of sets over α.

Suppose $\mathbf{D}_o$ and $y_{o\alpha} \notin \mathbf{Free}(\mathbf{D})$.

Comprehension Principle for $\mathbf{D}$:

$$\exists y_{o\alpha} \forall x_\alpha . [y\, x] \equiv \mathbf{D}_o$$

The term $[\lambda x_\alpha \mathbf{D}]$ witnesses the Comprehension Principle.

This works for any arity. $(o\alpha^n \cdots \alpha^1)$ is a type of relations.
For any $\mathbf{D}_o$ and $y_{o\alpha^n \cdots \alpha^1} \notin \mathbf{Free}(\mathbf{D})$, $[\lambda x^1 \cdots \lambda x^n \mathbf{D}]$ witnesses:

$$\exists y_{o\alpha^n \cdots \alpha^1} \forall x^1_{\alpha^1} \ldots \forall x^n_{\alpha^n} . [y\, x^1 \cdots x^n] \equiv \mathbf{D}_o$$

Conclusion: λ-terms $\Rightarrow$ Comprehension.

Surjective Cantor Theorem

There is no surjection from $\mathcal{D}_\iota$ onto $\mathcal{D}_{o\iota}$.

$$\neg\exists\, g_{o\iota\iota}\, \forall\, f_{o\iota}\, \exists\, x_\iota\, \mathbf{.}\, g\, x \;=^{o\iota}\; f$$

Suppose $G_{o\iota\iota}$ is a surjection.

- Let $D_{o\iota}$ be $[\lambda x_\iota\, \neg\mathbf{.}\, G\, x\, x]$ (diagonal set).

$$\{x \,\Big|\, x \notin [G\, x]\}$$

- There is some X_ι such that $G\, X \;=^{o\iota}\; D$.

$$G\, X\, X \;\equiv\; D\, X \;\equiv\; \neg\mathbf{.}\, G\, X\, X$$

Contradiction

Instantiating Set Variables

When a program (Tps) searches for a proof of a theorem, we may need to instantiate for a set variable using logical constants. The diagonal set $[\lambda x_\iota \neg . G\, x\, x]$ (using $\neg$) is a simple example.

Questions:

- Do we need to consider *every* logical constant?

What about Π^o or Π^{oo}?

Do we ever need an instantiation like $[\lambda x_\alpha \forall y_o \forall z_{oo}\, \mathbf{A}_o]$?

- Can we assume instantiations are in a normal form?

Do we ever need an instantiation like $[\lambda x_\alpha \neg\neg \mathbf{A}_o]$?

- Can we avoid instantiating some set variables altogether?

Leibniz equality: $\forall q_{o\alpha} . [q\, \mathbf{A}_\alpha] \supset [q\, \mathbf{B}_\alpha]$.

Instantiating Set Variables

Question:

- Do we need to consider *every* logical constant?

What logical constants are necessary for completeness?

We concentrate now on this question.

Surjective Cantor Theorem

Defining diagonal set $[\lambda x_\iota \neg . G\, x\, x]$ required $\neg$.

We used the logical constants $\neg$, $\Sigma^{o\iota\iota}$, $\Pi^{o\iota}$, Σ^{ι} and $=^{o\iota}$ to write

$$\neg\exists\, g_{o\iota\iota} \forall\, f_{o\iota} \exists\, x_\iota . g\, x \; =^{o\iota} \; f$$

as a term. ($[\exists\, x_\iota \cdots]$ means $[\Sigma^\iota\, \lambda x_\iota \cdots]$).

What if $\neg$ is not included as a logical constant?

Can we express the theorem without negation?

Yes, as a proposition *over* terms:

$$\dot{\neg}\, \dot{\exists}\, g_{o\iota\iota}\, \dot{\forall}\, f_{o\iota}\, \dot{\exists}\, x_\iota\, [[g\, x] \;\dot{=}^{o\iota}\; f]$$

Is this provable without negation? No.

Comprehension Principles for Propositions

Suppose $\mathbf{M}$ is a proposition and $y_{o\alpha^n\cdots\alpha^1} \notin \mathbf{Free}(\mathbf{M})$.

Is there a witness of the Comprehension Principle for $\mathbf{M}$?

$$\dot{\exists} y_{o\alpha^n\cdots\alpha^1} \, \dot{\forall} x^1_{\alpha^1} \ldots \dot{\forall} x^n_{\alpha^n} \centerdot [y \, x^1 \cdots x^n] \mathrel{\dot{\equiv}} \mathbf{M}$$

Example:

$$\dot{\exists} y_{o\iota} \, \dot{\forall} x_\iota \centerdot [y \, x] \mathrel{\dot{\equiv}} \dot{\neg} [G_{o\iota\iota} \, x \, x]$$

The diagonal set $[\lambda x_\iota \neg [G \, x \, x]]$ witnesses this *assuming* $\neg \in \mathcal{S}$.

Logical constants in $\mathcal{S}$ correspond to Comprehension Principles for Propositions.

A Worthless Construction

Claim: There is a "model" where Surjective Cantor fails.

Define $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}_\iota := (-1, 1)$ (Real Interval).

Define $\mathcal{D}_{o\iota} := \{\chi_{(a,1)} : (-1, 1) \to \{\mathtt{T}, \mathtt{F}\} \,\big|\, -1 \leq a \leq 1\} \subseteq (\mathcal{D}_o)^{\mathcal{D}_\iota}$.

Define $\mathcal{D}_{\alpha\beta} \subseteq (\mathcal{D}_\alpha)^{\mathcal{D}_\beta}$ arbitrarily for other types $\alpha\beta$.

There is a surjection from $\mathcal{D}_\iota$ to $\mathcal{D}_{o\iota}$ (same cardinality).

This Proves Nothing! Two Reasons:

- There's no interpretation for arbitrary λ-terms.
- This is an argument ala **Skolem's "Paradox"**.

We need a surjection $g : \mathcal{D}_\iota \to \mathcal{D}_{o\iota}$ with $\boxed{g \in \mathcal{D}_{o\iota\iota}}$.

Frames in General

$$\begin{array}{lcl} \mathcal{D}_o & \subseteq & \{\mathtt{T}, \mathtt{F}\} \text{ (nonempty)} \\ \mathcal{D}_\iota & = & \text{any nonempty set} \\ \mathcal{D}_{\alpha\beta} & \subseteq & (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} \text{ (maybe not all functions)} \end{array}$$

To interpret λ-abstractions, $\mathcal{D}_{\alpha\beta}$ must contain "enough" functions from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$. Such frames are combinatory.

Examples:

- $[\lambda y_\beta\, A_\alpha]_{\alpha\beta}$ for any parameter A_α:
All constant functions must be in $\mathcal{D}_{\alpha\beta}$.

- $[\lambda x_\alpha\, x]_{\alpha\alpha}$: Identity function must be in $\mathcal{D}_{\alpha\alpha}$.

Frames Possessing Logical Constants

$$\begin{array}{lcl} \mathcal{D}_o & \subseteq & \{\mathtt{T},\mathtt{F}\} \text{ (nonempty)} \\ \mathcal{D}_\iota & = & \text{any nonempty set} \\ \mathcal{D}_{\alpha\beta} & \subseteq & (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} \text{ (maybe not all functions)} \end{array}$$

We may want $\mathcal{D}$ to include interpretations for logical constants.

• We say $\mathcal{D}$ possesses a logical constant c_α if there is an *appropriate* interpretation for c in $\mathcal{D}_\alpha$.

Examples:

• $\mathcal{D}$ possesses $\top$ if $\mathtt{T} \in \mathcal{D}_o$.

• $\mathcal{D}$ possesses $\bot$ if $\mathtt{F} \in \mathcal{D}_o$.

• $\mathcal{D}$ possesses $\neg$ if the negation function $\neg : \mathcal{D}_o \longrightarrow \mathcal{D}_o$ is in $\mathcal{D}_{oo}$.

• $\mathcal{D}$ possesses Π^α if there is some function $\Pi^\alpha \in \mathcal{D}_{o(o\alpha)}$ such that $\Pi^\alpha(f) = \mathtt{T}$ iff $f(a) = \mathtt{T}$ for every $a \in \mathcal{D}_\alpha$.

Interpretations of Logical Constants in Frames

$\ldots$ $\ldots$

$\Pi^o, \Sigma^o \in^? \mathcal{D}_{o(oo)}$ $\Pi^\iota, \Sigma^\iota \in^? \mathcal{D}_{o(o\iota)}$ $\ldots$ $\mathcal{D}_{o(\iota\iota)}$ $\mathcal{D}_{\iota(\iota\iota)}$

$\wedge, \vee, \equiv \in^? \mathcal{D}_{ooo}$ $\mathcal{D}_{\iota(o\iota)}$ $=^\iota \in^? \mathcal{D}_{o\iota\iota}$ $\mathcal{D}_{\iota\iota\iota}$

$\neg \in^? \mathcal{D}_{oo}$ $\mathcal{D}_{o\iota}$ $\mathcal{D}_{\iota o}$ $\mathcal{D}_{\iota\iota}$

$\mathtt{T}, \mathtt{F} \in^? \mathcal{D}_o$ $\mathcal{D}_\iota$

$\mathcal{S}$-Models

$$
\begin{array}{lcl}
\mathcal{D}_o & \subseteq & \{\mathtt{T}, \mathtt{F}\} \text{ (nonempty)} \\
\mathcal{D}_\iota & = & \text{any nonempty set} \\
\mathcal{D}_{\alpha\beta} & \subseteq & (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} \text{ (maybe not all functions)}
\end{array}
$$

Let $\mathcal{S}$ be a signature of logical constants.

- We say $\mathcal{D}$ possesses $\mathcal{S}$ if $\mathcal{D}$ possesses every $c \in \mathcal{S}$.
- An $\mathcal{S}$-model (of the $\mathcal{S}$ fragment of extensional type theory) is a combinatory frame $\mathcal{D}$ possessing $\mathcal{S}$.

There is a proof system $\vdash_{\mathcal{S}} \mathbf{M}$ for closed propositions $\mathbf{M}$.

$\mathcal{S}$-models are sound and complete with respect to $\vdash_{\mathcal{S}}$.

Conservation

Suppose $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

Defn: A signature $\mathcal{S}_2$ is conservative over $\mathcal{S}_1$ if $\vdash_{\mathcal{S}_1} \mathbf{M}$ whenever $\vdash_{\mathcal{S}_2} \mathbf{M}$ and $\mathbf{M}$ is expressible using $\mathcal{S}_1$.

- If $\mathcal{S}_2$ is *not* conservative over $\mathcal{S}_1$, then there is some proposition $\mathbf{M}$ with $\vdash_{\mathcal{S}_2} \mathbf{M}$ but $\not\vdash_{\mathcal{S}_1} \mathbf{M}$.
- By completeness, there is some $\mathcal{S}_1$-model $\mathcal{D}$ where $\mathbf{M}$ is false.
- By soundness, $\mathcal{D}$ cannot be an $\mathcal{S}_2$-model.
- There is some logical constant $c \in \mathcal{S}_2 \setminus \mathcal{S}_1$ not possessed by $\mathcal{D}$.

Conservation and Independence

Suppose $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

- We can show $\mathcal{S}_2$ is conservative over $\mathcal{S}_1$ by showing every $\mathcal{S}_1$-model possess $\mathcal{S}_2$.

- We can show $\mathcal{S}_2$ is *not* conservative over $\mathcal{S}_1$ by constructing an $\mathcal{S}_1$-model which does not possess some $c \in \mathcal{S}_2 \setminus \mathcal{S}_1$.

Conservation (Propositional Constants)

Let $\mathcal{D}$ be an $\mathcal{S}$-model where $\neg, \vee \in \mathcal{S}$. Then:

- $\mathcal{D}$ must possess $\{\top, \bot, \wedge, \supset, \equiv\}$ (definable from $\neg$ and $\vee$).
- $\mathcal{D}$ must possess $\{\Pi^\beta, \Sigma^\beta, =^\beta \;\big|\; \beta \in \mathcal{T}_o\}$ where

$\mathcal{T}_o = \{o, (oo), (ooo), (o(oo)), \ldots\}$ (propositional types).

- $\mathcal{S} \cup \{\top, \bot, \wedge, \equiv\} \cup \{\Pi^\beta, \Sigma^\beta, =^\beta \;\big|\; \beta \in \mathcal{T}_o\}$ is conservative over $\mathcal{S}$.

Conclusion:

- If we consider set instantiations using $\neg$ and $\vee$, we need not consider set instantiations using constants from the set

$$\{\top, \bot, \wedge, \supset, \equiv\} \cup \{\Pi^\beta, \Sigma^\beta, =^\beta \;\big|\; \beta \in \mathcal{T}_o\}$$

Conservation (Quantifiers and Equality)

• If a model $\mathcal{D}$ possesses Π^{β} and $=^{\alpha}$, then $\mathcal{D}$ must possess $=^{\alpha\beta}$.
Extensional Definition:

$$[\lambda f_{\alpha\beta}\, \lambda g_{\alpha\beta} \forall y_{\beta} \centerdot f\, y \;=^{\alpha}\; g\, y]$$

• If a model $\mathcal{D}$ possesses $=^{o\alpha}$ and $\top$, then $\mathcal{D}$ must possess Π^{α}.
Definition:

$$[\lambda p_{o\alpha} \centerdot p \;=^{o\alpha}\; \lambda x_{\alpha} \top]$$

Combinatory Frames (Independence)

Suppose $\mathcal{D}$ is a combinatory frame and $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$.

There are four functions from $\mathcal{D}_o$ to $\mathcal{D}_o$: two constant functions, identity, negation.

$[\lambda y_o\, A_o]_{oo}$ for any parameter A_o:
The constant functions $K_{\mathtt{F}}$ and $K_{\mathtt{T}}$ must be in $\mathcal{D}_{oo}$.

$[\lambda x_o\, x]_{oo}$: Identity function must be in $\mathcal{D}_{oo}$.

What about negation?

Combinatory Frame Without Negation

(Ignore type ι)

Define $\mathcal{D}_o := \{\mathtt{F}, \mathtt{T}\}$.
Order $\mathcal{D}_o$ by $\mathtt{F} <^o \mathtt{T}$.

Extend to function types:

Assume $\mathcal{D}_\alpha$ is defined and ordered by $\leq^\alpha$.
Assume $\mathcal{D}_\beta$ is defined and ordered by $\leq^\beta$.

Define $\mathcal{D}_{\alpha\beta}$ to be all monotone functions $f : \mathcal{D}_\beta \rightarrow \mathcal{D}_\alpha$.

$$\forall x, y \in \mathcal{D}_\beta . x \leq^\beta y \Rightarrow f(x) \leq^\alpha f(y)$$

Combinatory Frame Without Negation

Order $\mathcal{D}_{\alpha\beta}$ by $\leq^{\alpha\beta}$: $f \leq^{\alpha\beta} g$ iff

$$\forall x, y \in \mathcal{D}_{\beta} \mathbf{.} \, x \leq^{\beta} y \Rightarrow f(x) \leq^{\alpha} g(y).$$

The frame $\mathcal{D}$ is combinatory.

$\neg : \mathcal{D}_o \to \mathcal{D}_o$ is not monotone since $\mathtt{F} \leq^o \mathtt{T}$, but $\neg(\mathtt{F}) \not\leq^o \neg(\mathtt{T})$

Hence $\mathcal{D}_{oo} = \{K_{\mathtt{F}}, K_{\mathtt{T}}, id\}$.

(Binary) Logical Relation Frames

Start with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota$ nonempty,

$\mathcal{R}_o \subseteq (\mathcal{D}_o)^2$ (e.g., $\leq^o$) and $\mathcal{R}_\iota \subseteq (\mathcal{D}_\iota)^2$.

Assume $\mathcal{R}_o$ and $\mathcal{R}_\iota$ are reflexive binary relations.

Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^2$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^2$ are defined.

Define $\mathcal{D}_{\alpha\beta}$:

$$\{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \,\big|\, \forall \langle y^0, y^1 \rangle \in \mathcal{R}_\beta \Rightarrow \langle f(y^0), f(y^1) \rangle \in \mathcal{R}_\alpha\}$$

Define $\mathcal{R}_{\alpha\beta} \subseteq (\mathcal{D}_{\alpha\beta})^2$:

$$\{\langle f^0, f^1 \rangle \,\big|\, \forall \langle y^0, y^1 \rangle \in \mathcal{R}_\beta \Rightarrow \langle f^0(y^0), f^1(y^1) \rangle \in \mathcal{R}_\alpha\}$$

Logical Relation Frames

$$
\begin{array}{cccccccccc}
 & & \ldots & & & & & \ldots & & \\
\hline
\mathcal{D}_{o(oo)} & \mathcal{D}_{o(o\iota)} & \ldots & \mathcal{D}_{o(\iota\iota)} & \mathcal{D}_{\iota(\iota\iota)} & \mathcal{R}_{o(oo)} & \mathcal{R}_{o(o\iota)} & \ldots & \mathcal{R}_{o(\iota\iota)} & \mathcal{R}_{\iota(\iota\iota)} \\
\mathcal{D}_{ooo} & \mathcal{D}_{\iota(o\iota)} & & \mathcal{D}_{o\iota\iota} & \mathcal{D}_{\iota\iota\iota} & \mathcal{R}_{ooo} & \mathcal{R}_{\iota(o\iota)} & & \mathcal{R}_{o\iota\iota} & \mathcal{R}_{\iota\iota\iota} \\
\hline
\mathcal{D}_{oo} & \mathcal{D}_{o\iota} & \mathcal{D}_{\iota o} & \mathcal{D}_{\iota\iota} & & \mathcal{R}_{oo} & \mathcal{R}_{o\iota} & \mathcal{R}_{\iota o} & \mathcal{R}_{\iota\iota} & \\
\hline
\mathcal{D}_{o} & & \mathcal{D}_{\iota} & & & \mathcal{R}_{o} & & \mathcal{R}_{\iota} & &
\end{array}
$$

Given: $\mathcal{D}_o$, $\mathcal{D}_\iota$, $\mathcal{R}_o$, $\mathcal{R}_\iota$

Logical Relation Frames

$$\cdots \qquad\qquad\qquad\qquad \cdots$$

$$\mathcal{D}_{o(oo)} \quad \mathcal{D}_{o(o\iota)} \quad \cdots \quad \mathcal{D}_{o(\iota\iota)} \quad \mathcal{D}_{\iota(\iota\iota)} \quad \mathcal{R}_{o(oo)} \quad \mathcal{R}_{o(o\iota)} \quad \cdots \quad \mathcal{R}_{o(\iota\iota)} \quad \mathcal{R}_{\iota(\iota\iota)}$$

$$\mathcal{D}_{ooo} \quad \mathcal{D}_{\iota(o\iota)} \quad \mathcal{D}_{o\iota\iota} \quad \mathcal{D}_{\iota\iota\iota} \quad \mathcal{R}_{ooo} \quad \mathcal{R}_{\iota(o\iota)} \quad \mathcal{R}_{o\iota\iota} \quad \mathcal{R}_{\iota\iota\iota}$$

$$\mathcal{D}_{oo} \quad \mathcal{D}_{o\iota} \quad \mathcal{D}_{\iota o} \quad \mathcal{D}_{\iota\iota} \quad \mathcal{R}_{oo} \quad \mathcal{R}_{o\iota} \quad \mathcal{R}_{\iota o} \quad \mathcal{R}_{\iota\iota}$$

$$\mathcal{D}_{o} \qquad \mathcal{D}_{\iota} \qquad \mathcal{R}_{o} \qquad \mathcal{R}_{\iota}$$

Define: $\mathcal{D}_{\alpha\beta},\ \alpha, \beta \in \{o, \iota\}$

Logical Relation Frames

$\dots \qquad\qquad\qquad\qquad \dots$

$\mathcal{D}_{o(oo)}\ \mathcal{D}_{o(o\iota)}\ \dots\ \mathcal{D}_{o(\iota\iota)}\ \mathcal{D}_{\iota(\iota\iota)}\ \mathcal{R}_{o(oo)}\ \mathcal{R}_{o(o\iota)}\ \dots\ \mathcal{R}_{o(\iota\iota)}\ \mathcal{R}_{\iota(\iota\iota)}$

$\mathcal{D}_{ooo}\ \mathcal{D}_{\iota(o\iota)}\ \mathcal{D}_{o\iota\iota}\ \mathcal{D}_{\iota\iota\iota}\ \mathcal{R}_{ooo}\ \mathcal{R}_{\iota(o\iota)}\ \mathcal{R}_{o\iota\iota}\ \mathcal{R}_{\iota\iota\iota}$

$\mathcal{D}_{oo}\ \mathcal{D}_{o\iota}\ \mathcal{D}_{\iota o}\ \mathcal{D}_{\iota\iota}\ \mathcal{R}_{oo}\ \mathcal{R}_{o\iota}\ \mathcal{R}_{\iota o}\ \mathcal{R}_{\iota\iota}$

$\mathcal{D}_{o}\ \mathcal{D}_{\iota}\ \mathcal{R}_{o}\ \mathcal{R}_{\iota}$

Define: $\mathcal{R}_{\alpha\beta},\ \alpha, \beta \in \{o, \iota\}$

Logical Relation Frames

$$\ldots \qquad\qquad\qquad\qquad \ldots$$

$$\mathcal{D}_{o(oo)}\ \mathcal{D}_{o(o\iota)}\ \ldots\ \mathcal{D}_{o(\iota\iota)}\ \mathcal{D}_{\iota(\iota\iota)}\ \mathcal{R}_{o(oo)}\ \mathcal{R}_{o(o\iota)}\ \ldots\ \mathcal{R}_{o(\iota\iota)}\ \mathcal{R}_{\iota(\iota\iota)}$$

$$\mathcal{D}_{ooo}\ \mathcal{D}_{\iota(o\iota)}\ \mathcal{D}_{o\iota\iota}\ \mathcal{D}_{\iota\iota\iota}\ \mathcal{R}_{ooo}\ \mathcal{R}_{\iota(o\iota)}\ \mathcal{R}_{o\iota\iota}\ \mathcal{R}_{\iota\iota\iota}$$

$$\mathcal{D}_{oo}\quad \mathcal{D}_{o\iota}\quad \mathcal{D}_{\iota o}\quad \mathcal{D}_{\iota\iota}\quad \mathcal{R}_{oo}\quad \mathcal{R}_{o\iota}\quad \mathcal{R}_{\iota o}\quad \mathcal{R}_{\iota\iota}$$

$$\mathcal{D}_{o}\qquad \mathcal{D}_{\iota}\qquad \mathcal{R}_{o}\qquad \mathcal{R}_{\iota}$$

Define: $\mathcal{D}_{\alpha\beta}$ at next level

Logical Relation Frames

$$\ldots \qquad\qquad\qquad \ldots$$

$$\mathcal{D}_{o(oo)} \quad \mathcal{D}_{o(o\iota)} \quad \ldots \quad \mathcal{D}_{o(\iota\iota)} \quad \mathcal{D}_{\iota(\iota\iota)} \quad \mathcal{R}_{o(oo)} \quad \mathcal{R}_{o(o\iota)} \quad \ldots \quad \mathcal{R}_{o(\iota\iota)} \quad \mathcal{R}_{\iota(\iota\iota)}$$

$$\mathcal{D}_{ooo} \quad \mathcal{D}_{\iota(o\iota)} \quad \mathcal{D}_{o\iota\iota} \quad \mathcal{D}_{\iota\iota\iota} \quad \mathcal{R}_{ooo} \quad \mathcal{R}_{\iota(o\iota)} \quad \mathcal{R}_{o\iota\iota} \quad \mathcal{R}_{\iota\iota\iota}$$

$$\mathcal{D}_{oo} \quad \mathcal{D}_{o\iota} \quad \mathcal{D}_{\iota o} \quad \mathcal{D}_{\iota\iota} \quad \mathcal{R}_{oo} \quad \mathcal{R}_{o\iota} \quad \mathcal{R}_{\iota o} \quad \mathcal{R}_{\iota\iota}$$

$$\mathcal{D}_{o} \qquad \mathcal{D}_{\iota} \qquad \mathcal{R}_{o} \qquad \mathcal{R}_{\iota}$$

Define: $\mathcal{R}_{\alpha\beta}$ at next level

Logical Relation Frames

$\cdots \qquad \cdots$

$\mathcal{D}_{o(oo)}\ \mathcal{D}_{o(o\iota)}\ \cdots\ \mathcal{D}_{o(\iota\iota)}\ \mathcal{D}_{\iota(\iota\iota)}\ \mathcal{R}_{o(oo)}\ \mathcal{R}_{o(o\iota)}\ \cdots\ \mathcal{R}_{o(\iota\iota)}\ \mathcal{R}_{\iota(\iota\iota)}$

$\mathcal{D}_{ooo}\ \mathcal{D}_{\iota(o\iota)}\ \mathcal{D}_{o\iota\iota}\ \mathcal{D}_{\iota\iota\iota}\ \mathcal{R}_{ooo}\ \mathcal{R}_{\iota(o\iota)}\ \mathcal{R}_{o\iota\iota}\ \mathcal{R}_{\iota\iota\iota}$

$\mathcal{D}_{oo}\ \mathcal{D}_{o\iota}\ \mathcal{D}_{\iota o}\ \mathcal{D}_{\iota\iota}\ \mathcal{R}_{oo}\ \mathcal{R}_{o\iota}\ \mathcal{R}_{\iota o}\ \mathcal{R}_{\iota\iota}$

$\mathcal{D}_{o}\qquad \mathcal{D}_{\iota}\qquad \mathcal{R}_{o}\qquad \mathcal{R}_{\iota}$

$\cdots$ *and so on*

Logical Relation Frames

Fact 1: Each $\mathcal{R}_\alpha \subseteq \mathcal{D}_\alpha^2$ is reflexive.

Fact 2: $\mathcal{D}$ is a combinatory frame.

Construction works with relations of any arity.

Let A be a nonempty set.

function $p : A \to \mathcal{D}_\alpha$	$\sim$	A-tuple $\langle p(i) \rangle_{i \in A}$ of elements of $\mathcal{D}_\alpha$
set of functions $\mathcal{R}_\alpha \subseteq \mathcal{D}_\alpha^A$	$\sim$	A-ary relation on $\mathcal{D}_\alpha$

Logical Relation Frames

Let $x \in \mathcal{D}_\alpha$ and $K_x : A \to \mathcal{D}_\alpha$ be the

constant function $K_x(i) = x$ for all $i \in A$.

constant function K_x $\sim$ A-tuple $\langle x \rangle_{i \in A}$.

Instead of *reflexivity* of $\mathcal{R}_\alpha$,

we will need all constant functions K_x to be in $\mathcal{R}_\alpha$.

Generalizing Logical Relation Frames

Start with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota$ nonempty, $\mathcal{R}_o \subseteq \mathcal{D}_o^{\boxed{2}}$ and $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^{\boxed{2}}$.

Assume $\mathcal{R}_o$ and $\mathcal{R}_\iota$ include constant functions.

Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^{\boxed{2}}$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^{\boxed{2}}$ are defined.

Define $\mathcal{D}_{\alpha\beta} := \;\; \{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \mid \forall p \in \mathcal{R}_\beta \;\; \boxed{\boxed{\langle f(p^0), f(p^1)\rangle}} \in \mathcal{R}_\alpha\}$

Define $\mathcal{R}_{\alpha\beta} := \{\boxed{\boxed{q \in (\mathcal{D}_{\alpha\beta})^2}} \mid \forall p \in \mathcal{R}_\beta \;\; \boxed{\boxed{\langle q^0(p^0), q^1(p^1)\rangle}} \in \mathcal{R}_\alpha\}$

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Define $\mathcal{R}_{\alpha\beta} := \left\{ \boxed{q : \{0,1\} \to \mathcal{D}_{\alpha\beta}} \;\middle|\; \forall p \in \mathcal{R}_\beta \;\; \boxed{\langle q^i(p^i) \rangle_{i \in \{0,1\}}} \in \mathcal{R}_\alpha \right\}$

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Define $\mathcal{D}_{\alpha\beta} := \quad \{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \mid \forall p \in \mathcal{R}_\beta \ \boxed{(f \circ p)} \in \mathcal{R}_\alpha\}$

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where $S(q,p)(i) := q(i)(p(i))$.

Logical Relation Frames

Facts:

- Each $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^A$ contains all constant functions.
- $\mathcal{D}$ is a combinatory frame.

Frames with Specified Sets

Let A be nonempty and $\mathcal{B} \subseteq \mathcal{P}A$. Assume $\emptyset \in \mathcal{B}$ and $A \in \mathcal{B}$.

Define $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}_\iota := A$.

Define $\mathcal{R}_o \subseteq (\mathcal{D}_o)^A$: $\{p : A \rightarrow \mathcal{D}_o \,\big|\, p^{-1}(\mathtt{T}) \in \mathcal{B}\} = \{\chi_X \,\big|\, X \in \mathcal{B}\}$

Define $\mathcal{R}_\iota \subseteq (\mathcal{D}_\iota)^A$: $\{p : A \rightarrow \mathcal{D}_\iota \,\big|\, p^{-1}(X) \in \mathcal{B}$ whenever $X \in \mathcal{B}\}$

Extend $\mathcal{D}$ and $\mathcal{R}$ to all types giving combinatory frame $\mathcal{D}$.

Fact: $\mathcal{D}_{o\iota} = \{\chi_X | X \in \mathcal{B}\}$. $(\mathcal{D}_{o\iota} \cong \mathcal{B})$

Idea: We *specified* $\mathcal{D}_\iota$ and $\mathcal{D}_{o\iota}$ by giving A and $\mathcal{B}$,
then extended definition to all types using relations.

Frames with Specified Sets

$\ldots \qquad \ldots$

$\mathcal{D}_{o(oo)} \quad \mathcal{D}_{o(o\iota)} \quad \ldots \quad \mathcal{D}_{o(\iota\iota)} \quad \mathcal{D}_{\iota(\iota\iota)}$

$\mathcal{D}_{ooo} \quad \mathcal{D}_{\iota(o\iota)} \quad \mathcal{D}_{o\iota\iota} \quad \mathcal{D}_{\iota\iota\iota}$

$\mathcal{D}_{oo} \quad \mathcal{D}_{o\iota} \cong \mathcal{B} \quad \mathcal{D}_{\iota o} \quad \mathcal{D}_{\iota\iota}$

$\mathcal{D}_{o} = \{\mathtt{T}, \mathtt{F}\} \qquad \mathcal{D}_{\iota} = A$

Specified Sets: $\mathcal{D}_{\alpha\iota} \subseteq \mathcal{R}_\alpha$

$$\mathcal{D}_{\alpha\iota} \subseteq (\mathcal{D}_\alpha)^{\mathcal{D}_\iota} = (\mathcal{D}_\alpha)^A \supseteq \mathcal{R}_\alpha.$$

$$\mathcal{R}_\iota = \left\{p : A \to \mathcal{D}_\iota \,\middle|\, p^{-1}(X) \in \mathcal{B} \text{ whenever } X \in \mathcal{B}\right\}$$

$$\mathcal{D}_{\alpha\iota} = \left\{f : A \to \mathcal{D}_\alpha \,\middle|\, \forall p \in \mathcal{R}_\iota \ \ (f \circ p) \in \mathcal{R}_\alpha\right\}$$

Note: $id \in \mathcal{R}_\iota$.

For any $f \in \mathcal{D}_{\alpha\iota}$, $f = (f \circ id) \in \mathcal{R}_\alpha$.

$$\mathcal{D}_{\alpha\iota} \subseteq \mathcal{R}_\alpha$$

In particular,

$$\mathcal{D}_{o\iota} \subseteq \mathcal{R}_o$$

Every $f \in \mathcal{D}_{o\iota}$ is χ_X for some $X \in \mathcal{B}$.

Specified Sets: $\mathcal{R}_o \subseteq \mathcal{D}_{o\iota}$, $\mathcal{D}_{o\iota} \cong \mathcal{B}$

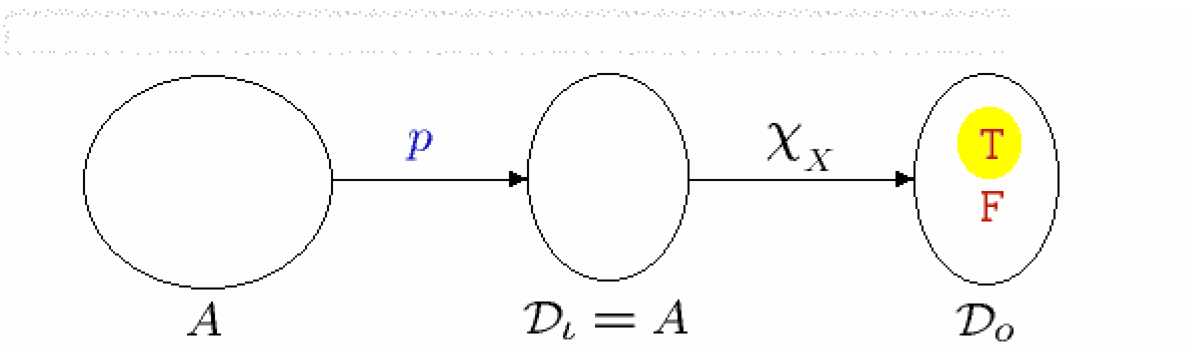
$$\mathcal{R}_o = \{p : A \to \mathcal{D}_o \;\Big|\; p^{-1}(\mathtt{T}) \in \mathcal{B}\}$$

$$\mathcal{R}_\iota = \{p : A \to \mathcal{D}_\iota \;\Big|\; p^{-1}(X) \in \mathcal{B} \text{ whenever } X \in \mathcal{B}\}$$

$$\mathcal{D}_{o\iota} = \{f : A \to \mathcal{D}_o \;\Big|\; \forall p \in \mathcal{R}_\iota \;\; (f \circ p) \in \mathcal{R}_o\}$$

Suppose $X \in \mathcal{B}$ and $p \in \mathcal{R}_\iota$. $(\chi_X \circ p) \in \mathcal{R}_o$?

Yes: $(\chi_X \circ p)^{-1}(\mathtt{T}) = p^{-1}(X) \in \mathcal{B}$ since $p \in \mathcal{R}_\iota$.



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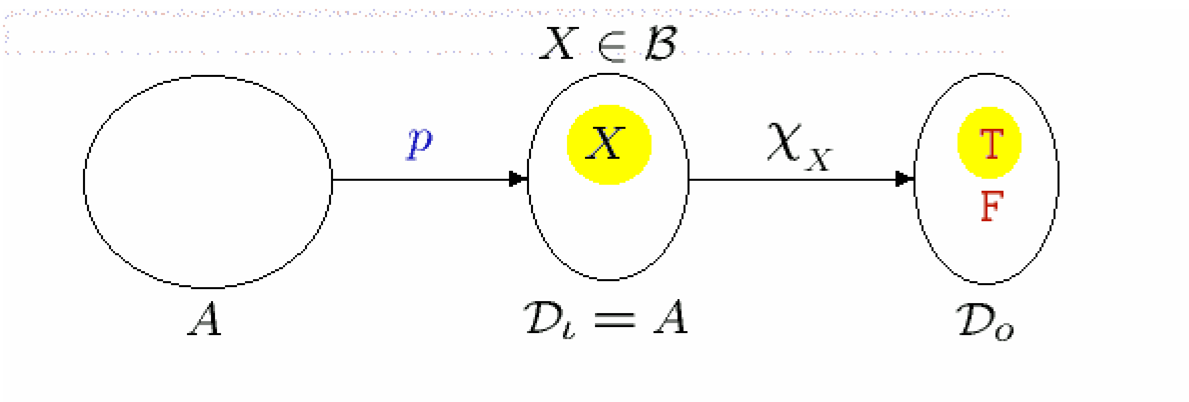

Specified Sets: $\mathcal{R}_o \subseteq \mathcal{D}_{o\iota}$, $\mathcal{D}_{o\iota} \cong \mathcal{B}$

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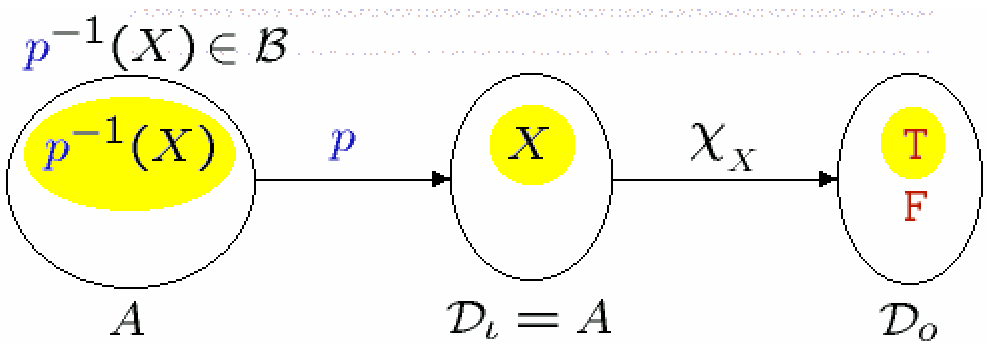


Hence $\mathcal{R}_o \subseteq \mathcal{D}_{o\iota}$ and $\mathcal{D}_{o\iota} = \mathcal{R}_o \cong \mathcal{B}$.

Specified Sets and Logical Constants

A nonempty, $\mathcal{B} \subseteq \mathcal{P}A$, $\emptyset \in \mathcal{B}$, $A \in \mathcal{B}$, $\mathcal{D}$ specified sets frame.

Let $\neg : \mathcal{D}_o \longrightarrow \mathcal{D}_o$ be the negation function. $\neg \in \mathcal{D}_{oo}$?

$\mathcal{R}_o = \{\chi_X | X \in \mathcal{B}\}$ $\qquad$ $\mathcal{D}_{oo} = \{f : \mathcal{D}_o \longrightarrow \mathcal{D}_o \,\big|\, \forall p \in \mathcal{R}_o \;\; (f \circ p) \in \mathcal{R}_o\}$

$$\begin{aligned} \neg \in \mathcal{D}_{oo} \quad & \text{iff} \quad (\neg \circ \chi_X) \in \mathcal{R}_o \text{ for all } X \in \mathcal{B} \\ & \text{iff} \quad \chi_{(A \setminus X)} \in \mathcal{R}_o \text{ for all } X \in \mathcal{B} \\ & \text{iff} \quad (A \setminus X) \in \mathcal{B} \text{ for all } X \in \mathcal{B} \end{aligned}$$

- $\mathcal{D}$ possesses $\neg$ iff $\mathcal{B}$ is closed under complements.

Specified Sets and Logical Constants

- $\mathcal{D}$ possesses $\wedge$ iff $\mathcal{B}$ is closed under binary intersections.
- $\mathcal{D}$ possesses $\vee$ iff $\mathcal{B}$ is closed under binary unions.

- We must directly check when $\mathcal{D}$ possesses $=^{\alpha}$.
- We must check when $\mathcal{D}$ possesses Π^{α} and Σ^{α}.

Frame in which Surjective Cantor Fails

Let A be the real interval $(-1, 1)$ and

$$\mathcal{B} := \{(a, 1) \mid -1 \leq a \leq 1\} \subseteq \mathcal{P}(A).$$

Note: $\emptyset$ is $(1, 1) \in \mathcal{B}$ and A is $(-1, 1) \in \mathcal{B}$.

Define $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}_\iota := A$.

Use Relations to define combinatory frame $\mathcal{D}$ with $\mathcal{D}_{o\iota} \cong \mathcal{B}$.

Facts:

- $\mathcal{D}_{o\iota} = \{\chi_{(a,1)} \mid -1 \leq a \leq 1\}$
- The negation function $\neg : \mathcal{D}_o \to \mathcal{D}_o$ is not in $\mathcal{D}_{oo}$.
- $\mathcal{D}$ possesses $\wedge$ and $\vee$.
- The Surjective Cantor Theorem fails.

Proposed Surjection in $\mathcal{D}_{o\iota\iota}$

$\mathcal{D}_\iota = (-1, 1)$

$\mathcal{D}_{o\iota} = \{\chi_{(a,1)} \mid -1 \leq a \leq 1\}$

Define $G : \mathcal{D}_\iota \to \mathcal{D}_{o\iota}$ as follows:

$$G(x) := \begin{cases} \chi_{\emptyset} & \text{if } -1 < x \leq -\frac{1}{2} \\ \chi_{(-2x,1)} & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ \chi_{(-1,1)} & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

That is, $G(x)(y) = \mathtt{T}$ iff $-2x < y$ for each $x, y \in (-1, 1)$.

Claim 1: G is surjective.

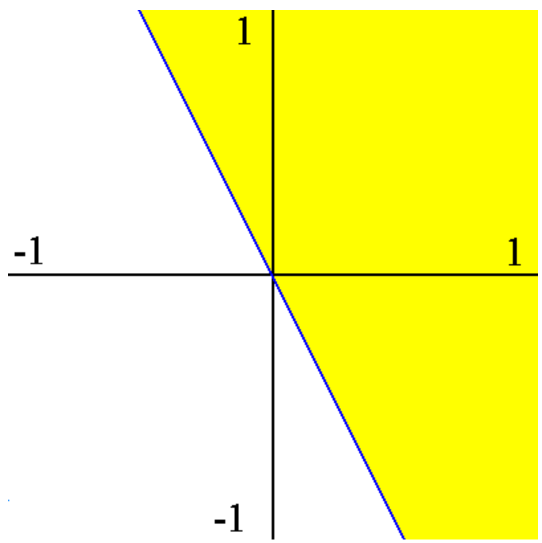
Claim 2: $G \in \mathcal{D}_{o\iota\iota}$.

Proposed Surjection in $\mathcal{D}_{o\iota\iota}$

$G(x)(y) = \mathtt{T}$ iff $-2x < y$ for each $x, y \in (-1, 1)$.

G is surjective since $G(-\frac{z}{2})$ is $\chi_{(z,1)}$ for any $(z, 1) \in \mathcal{B}$.

The graph of the relation $G(x)(y) = \mathtt{T}$:



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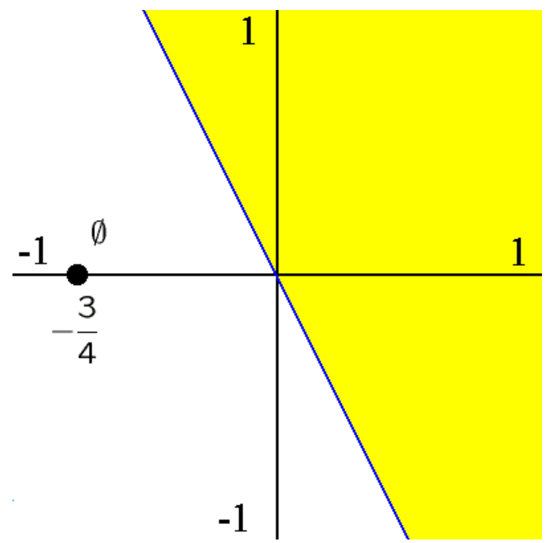

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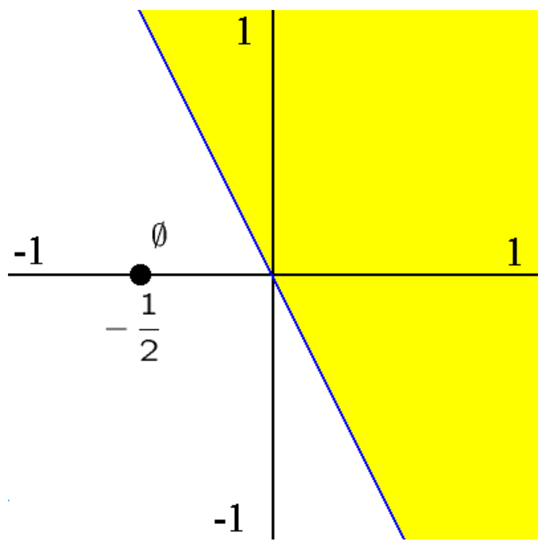

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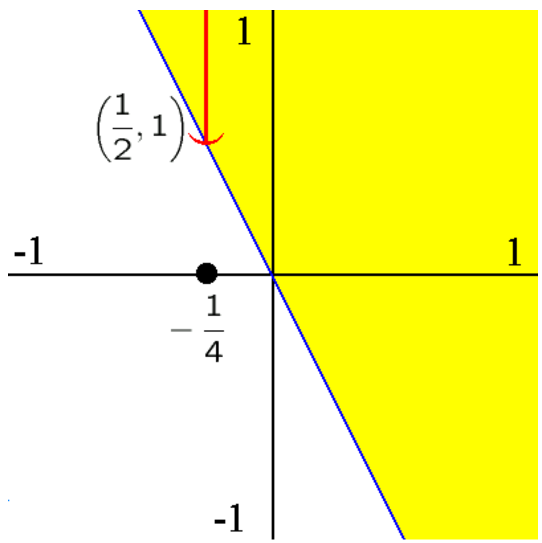

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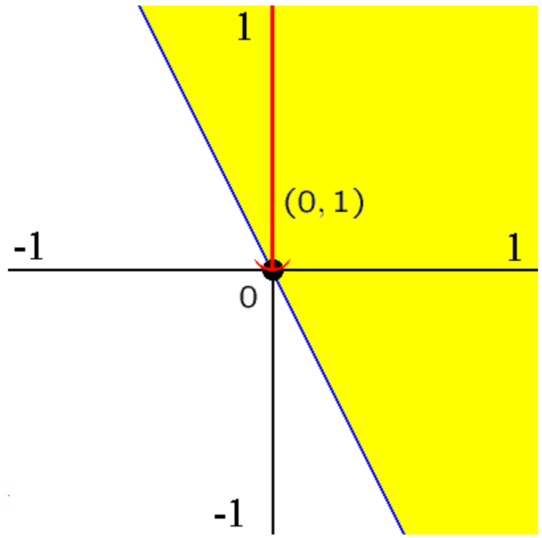

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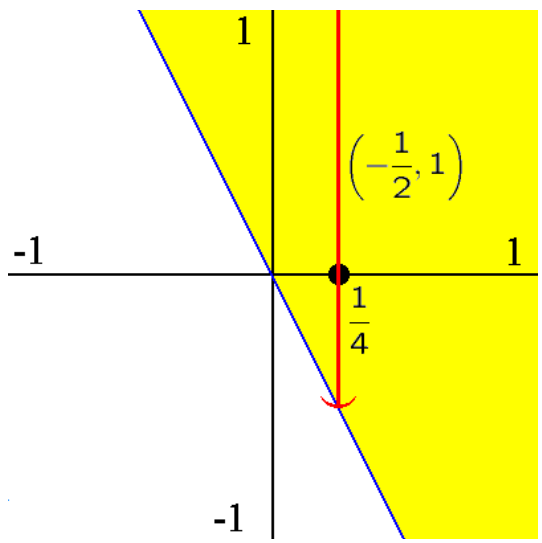

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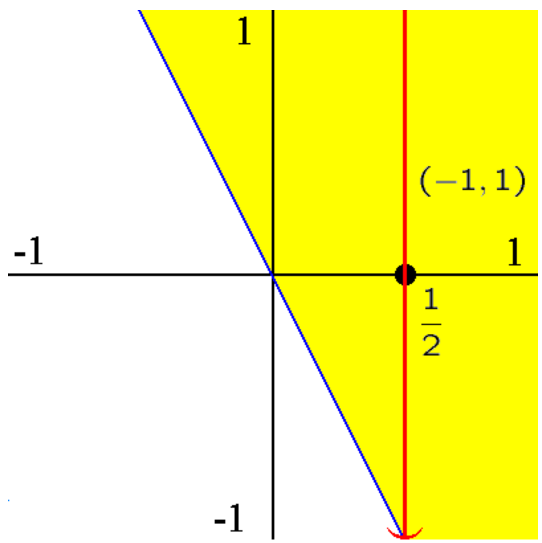

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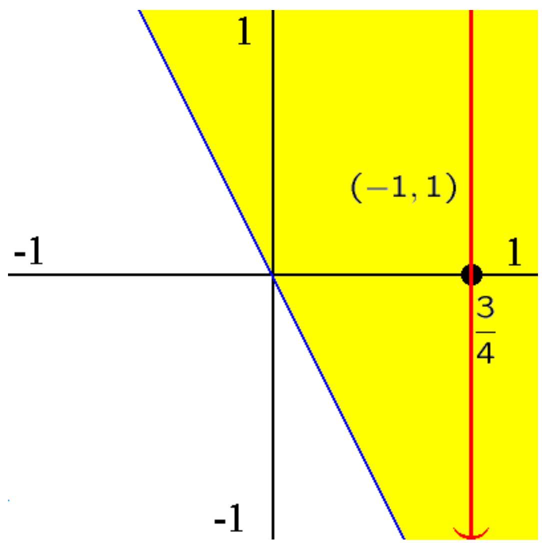


Proposed Surjection in $\mathcal{D}_{o\iota\iota}$

$G(x)(y) = \mathtt{T}$ iff $-2x < y$ for each $x, y \in (-1, 1)$.

Is $G \in \mathcal{D}_{o\iota\iota}$?

Need to check $(G \circ p_1) \in \mathcal{R}_{o\iota}$ for every $p_1 \in \mathcal{R}_{\iota}$.

To check $(G \circ p_1) \in \mathcal{R}_{o\iota}$, we need to check

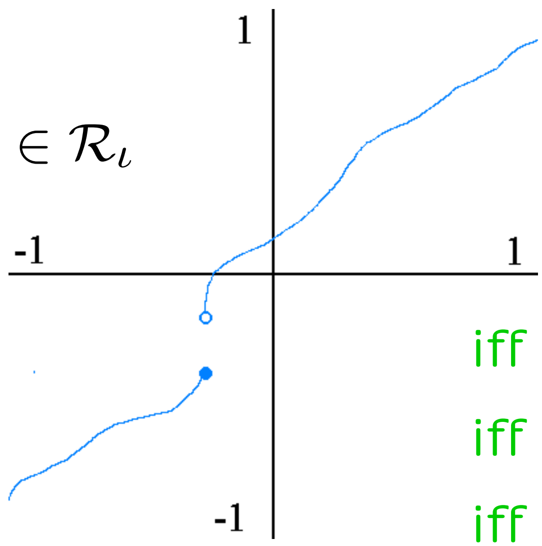
$S((G \circ p_1), p_2) \in \mathcal{R}_{o}$ for every $p_2 \in \mathcal{R}_{\iota}$.

First: Characterize $\mathcal{R}_{\iota}$.

Frame in which Surjective Cantor Fails: $\mathcal{R}_\iota$

$$\mathcal{R}_\iota = \{p : A \to \mathcal{D}_\iota \;\big|\; \forall X \in \mathcal{B} \;\; p^{-1}(X) \in \mathcal{B}\}$$

$p \in \mathcal{R}_\iota$



Suppose $p : A \to A$.

$p \in \mathcal{R}_\iota$

iff $p^{-1}(X) \in \mathcal{B}$ for every $X \in \mathcal{B}$

iff $p^{-1}((a, 1)) \in \mathcal{B}$ for every $a \in [-1, 1]$

iff p is nondecreasing and left-continuous.

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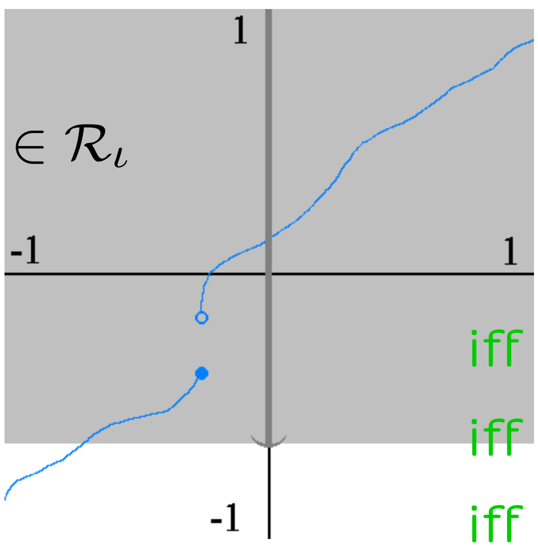

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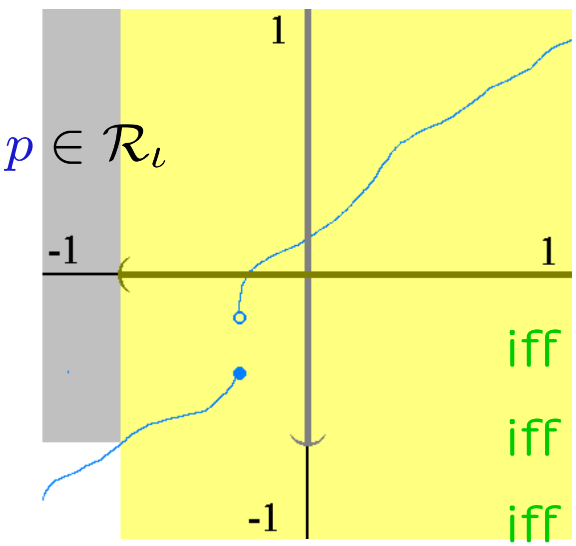

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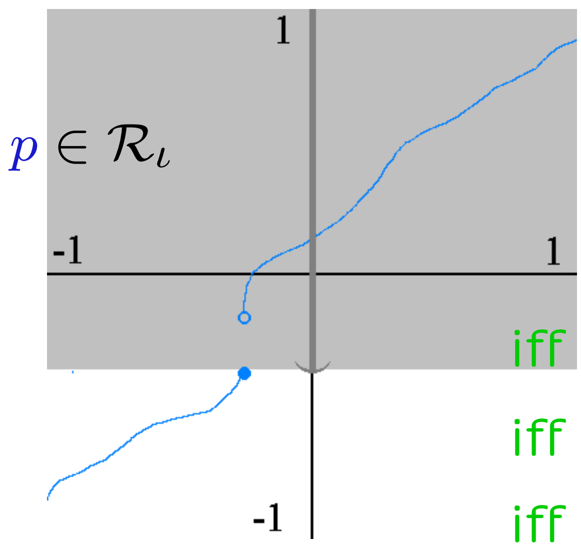

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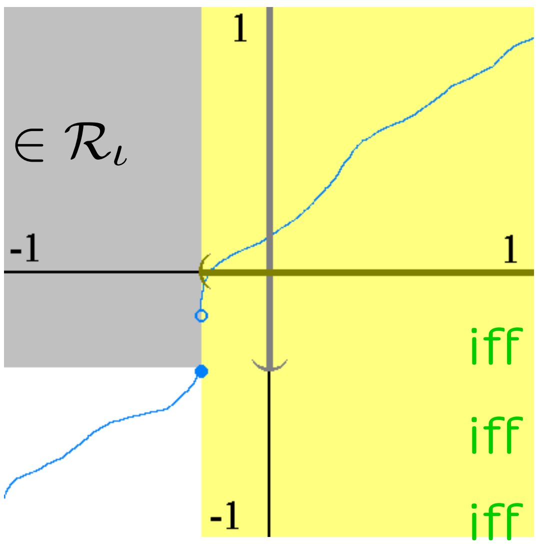

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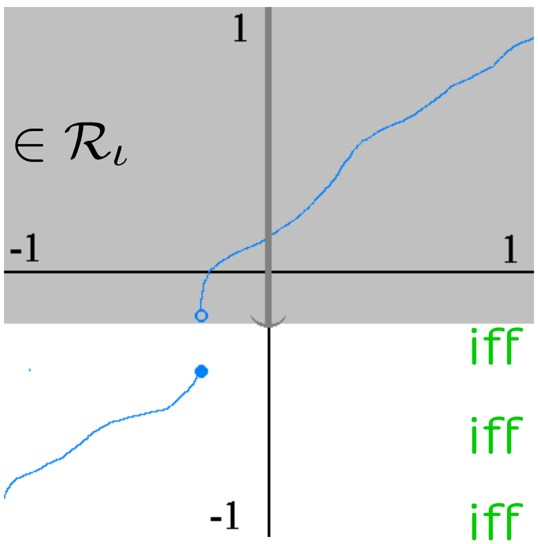

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iff $p^{-1}(X) \in \mathcal{B}$ for every $X \in \mathcal{B}$

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iff p is nondecreasing and left-continuous.

Frame in which Surjective Cantor Fails: $\mathcal{R}_\iota$

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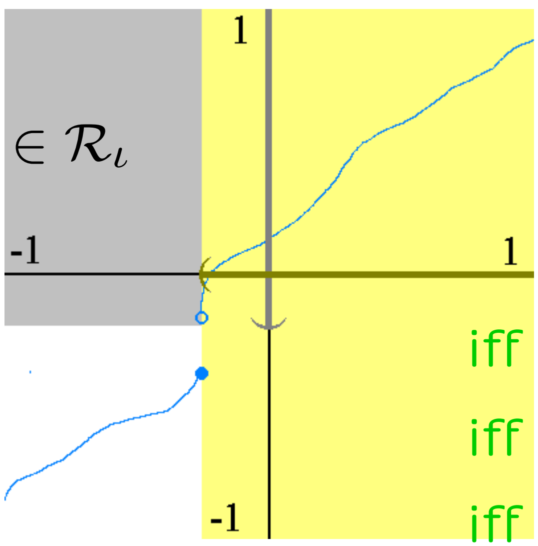

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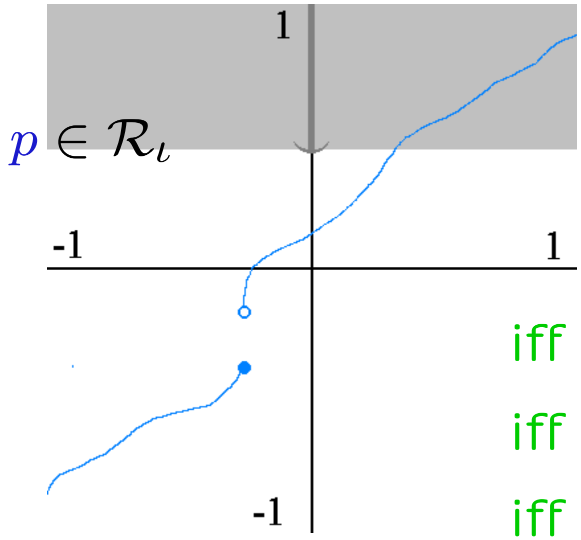

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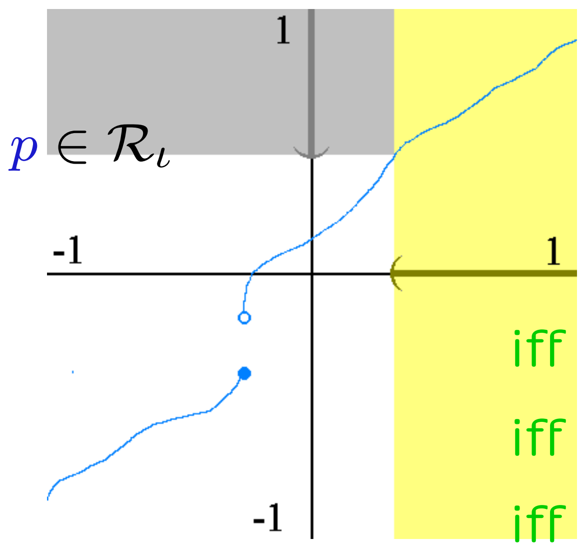

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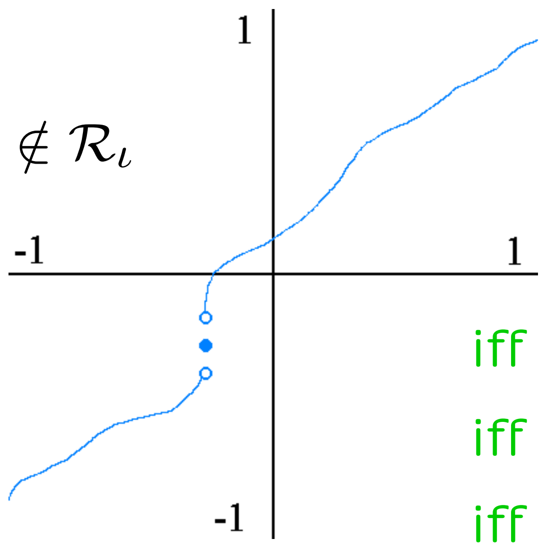

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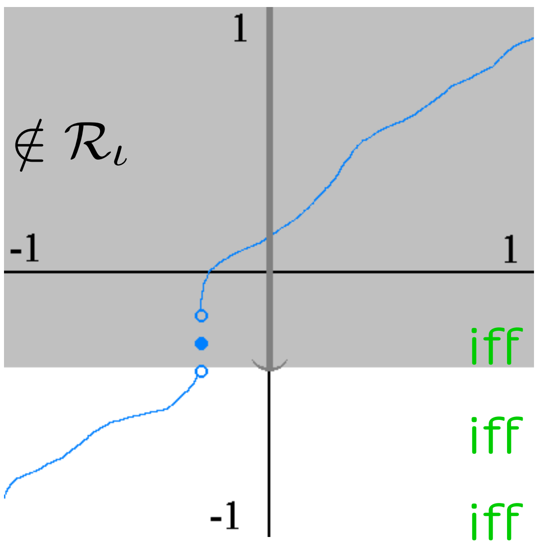

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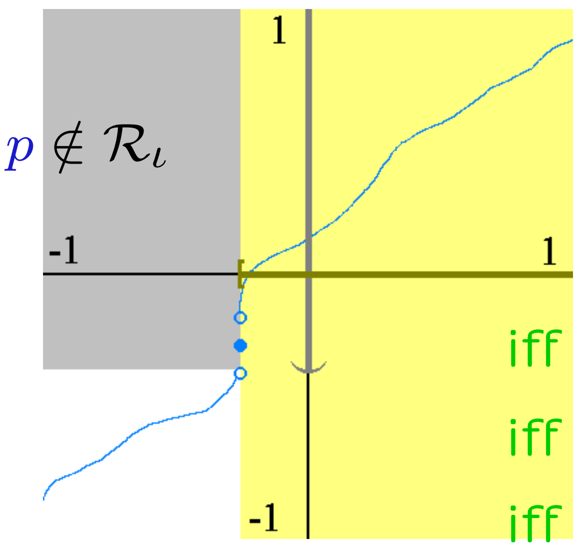

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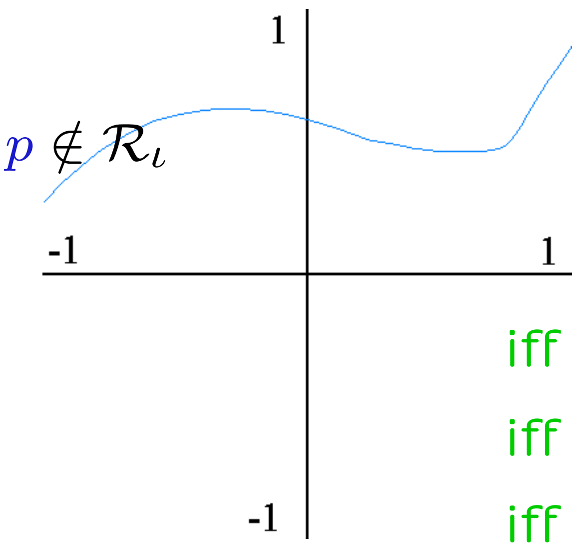

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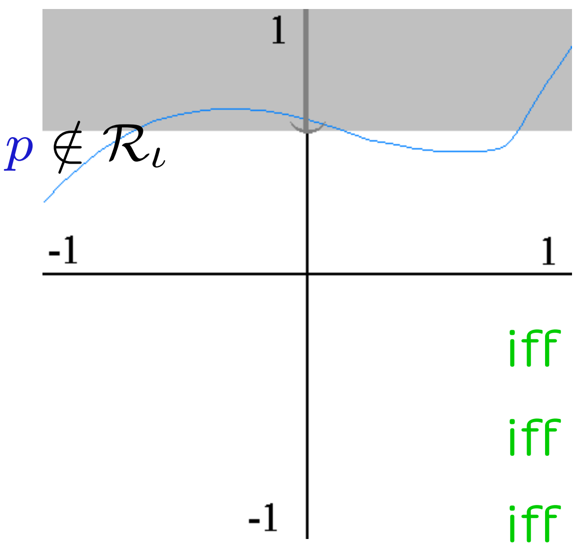

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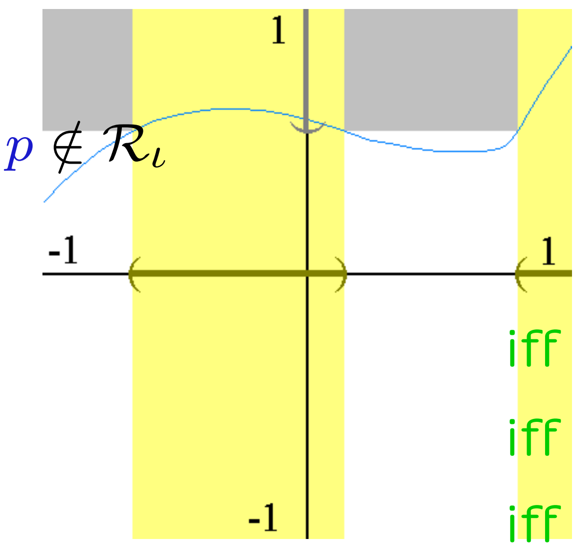

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Frame in which Surjective Cantor Fails

A is $(-1, 1)$ and $\mathcal{B}$ is $\{(a, 1) \mid -1 \leq a \leq 1\} \subseteq \mathcal{P}(A)$.

Is $G \in \mathcal{D}_{o\iota\iota}$?

Let $p_1, p_2 \in \mathcal{R}_\iota$.

$p_1, p_2 : (-1, 1) \to (-1, 1)$ are nondecreasing and left-continuous.

Show: $S((G \circ p_1), p_2) \in^? \mathcal{R}_o$.

Need: $S((G \circ p_1), p_2)^{-1}(\mathtt{T}) \in^? \mathcal{B}$.

$$\{x \in (-1, 1) \mid S((G \circ p_1), p_2)(x) = \mathtt{T}\} \in^? \mathcal{B}$$

Frame in which Surjective Cantor Fails

Show: $\{x \in (-1,1) \mid S((G \circ p_1), p_2)(x) = \mathtt{T}\} \in^? \mathcal{B}$

Compute:

$$S((G \circ p_1), p_2)(x) = (G \circ p_1)(x)(p_2(x)) = G(p_1(x))(p_2(x))$$

Show: $\{x \in (-1,1) \mid G(p_1(x))(p_2(x)) = \mathtt{T}\} \in^? \mathcal{B}$

Recall: $G(x)(y) = \mathtt{T}$ iff $-2x < y$ for each $x, y \in (-1,1)$.

Simplifies Using Definition of G:

$$\{x \in (-1,1) \mid -2p_1(x) < p_2(x)\} \in^? \mathcal{B}$$

Frame in which Surjective Cantor Fails

Show: $\{x \in (-1,1) \bigm| -2p_1(x) < p_2(x)\} \in^? \mathcal{B}$

Show: $\{x \in (-1,1) \bigm| 0 < (p_2 + 2p_1)(x)\} \in^? \mathcal{B}$

$(p_2 + 2p_1)$ is nondecreasing and left-continuous since $p_1, p_2 \in \mathcal{R}_\iota$ are nondecreasing and left-continuous.

Thus

$$\{x \in (-1,1) \bigm| 0 < (p_2 + 2p_1)(x)\} = (a, 1) \in \mathcal{B}$$

for some $a \in [-1, 1]$.

Therefore, the surjection G is in $\mathcal{D}_{o\iota\iota}$ as desired.

Injective Cantor Theorem

There is no injection from $\mathcal{D}_{o\iota}$ to $\mathcal{D}_{\iota}$.

$$\dot{\neg}\,\dot{\exists}\, h_{\iota(o\iota)} \dot{\forall}\, p_{o\iota} \dot{\forall}\, q_{o\iota} \mathbf{.}\, h\, p \doteq^{\iota} h\, q \;\dot{\supset}\; p \doteq^{o\iota} q$$

Informal Proof: Assume $H_{\iota(o\iota)}$ is injective.

- Let $\mathbf{D}$ be $\{[H\, X_{o\iota}] | [H\, X] \notin X\}$ (diagonal set).
- Consider $[H\, \mathbf{D}]$. If $[H\, \mathbf{D}] \notin \mathbf{D}$, then $[H\, \mathbf{D}] \in \mathbf{D}$. Hence $[H\, \mathbf{D}] \in \mathbf{D}$.
- For some $X_{o\iota}$, $[[H\, \mathbf{D}] =^{\iota} [H\, X]]$ and $[H\, X] \notin X$.
- By injectivity of H, $[X =^{o\iota} \mathbf{D}]$ and so $[H\, \mathbf{D}] \notin \mathbf{D}$.

Contradiction.

Injective Cantor Theorem

The diagonal set $\{[H\, X_{o\iota}] | [H\, X] \notin X\}$ can be formally defined as

$$[\lambda y_\iota \exists X_{o\iota} \centerdot [y =^\iota H\, X] \wedge \neg[X\, [H\, X]]]$$

if $\neg$, $\wedge$, $=^\iota$ and $\Sigma^{o\iota}$ are in the signature $\mathcal{S}$.

There is a combinatory frame $\mathcal{D}$ such that:

- The injective Cantor Theorem is false.
- The surjective Cantor Theorem is true.
- $\mathcal{D}$ possesses $\neg$ and $\wedge$.
- $\mathcal{D}$ possesses neither $=^\iota$ nor $\Sigma^{o\iota}$.

Frame in which Injective Cantor Fails

Let A be the natural numbers $\mathbb{N}$.

Let $\mathcal{B}$ be the set of finite $X \subseteq \mathbb{N}$ and cofinite $Y \subseteq \mathbb{N}$.

Note: $\emptyset \in \mathcal{B}$ (finite), $\mathbb{N} \in \mathcal{B}$ (cofinite), and $\mathcal{B}$ is closed under complements, binary unions and binary intersections.

Define $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}_\iota := \mathbb{N}$.

Use Relations to define combinatory frame $\mathcal{D}$ with $\mathcal{D}_{o\iota} \cong \mathcal{B}$.

$\mathcal{D}$ is a $\{\top, \bot, \neg, \wedge, \vee\}$-model.

Frame in which Injective Cantor Fails

$\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$

$\mathcal{D}_\iota = \mathbb{N}$

$\mathcal{R}_o = \{\chi_X : \mathbb{N} \to \mathcal{D}_o \mid X \text{ finite or cofinite}\}$

Fact: $p \in \mathcal{R}_o$ iff $p : \mathbb{N} \to \mathcal{D}_o$ is eventually constant.

Defn: A function $f : \mathbb{N} \to C$ is eventually constant if there is some $N \in \mathbb{N}$ and $c \in C$ such that $f(n) = c$ for every $n \geq N$.

Fact: For any type α, if $p : \mathbb{N} \to \mathcal{D}_\alpha$ is eventually constant, then $p \in \mathcal{R}_\alpha$.

Frame in which Injective Cantor Fails

Fact: If every $p \in \mathcal{R}_\beta$ is eventually constant, then $\mathcal{D}_{\alpha\beta} = (\mathcal{D}_\alpha)^{\mathcal{D}_\beta}$ for every type α.

Fact: Every $p \in \mathcal{R}_{o\iota}$ is eventually constant.

Conclusions:

- $f \in \mathcal{D}_{\iota(o\iota)}$ for every $f : \mathcal{D}_{o\iota} \to \mathcal{D}_\iota$.
- Every injection from $\mathcal{D}_{o\iota}$ to $\mathcal{D}_\iota$ (both are countable) is in $\mathcal{D}_{\iota(o\iota)}$.
- The Injective Cantor Theorem fails in $\mathcal{D}$.
- $f \in \mathcal{D}_{o(o\iota)}$ for every $f : \mathcal{D}_{o\iota} \to \mathcal{D}_o$.
- Π^ι and Σ^ι have interpretations in $\mathcal{D}_{o(o\iota)}$.

Contravariant Effects

- If $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^A$ is **small**, then the sets

$$\mathcal{D}_{o\alpha} := \{ f : \mathcal{D}_\alpha \to \mathcal{D}_o \,\big|\, \forall p \in \boxed{\mathcal{R}_\alpha} \;\; (f \circ p) \in \mathcal{R}_o \}$$

and

$$\mathcal{R}_{o\alpha} := \{ q : A \to \mathcal{D}_{o\alpha} \,\big|\, \forall p \in \boxed{\mathcal{R}_\alpha} \;\; S(f, p) \in \mathcal{R}_o \}$$

are **big** (fewer restrictions).

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Contravariant Effects

- $\mathcal{R}_\iota$ is **big**.
- $\mathcal{D}_{o\iota}$ and $\mathcal{R}_{o\iota}$ are **small**.
- $\mathcal{D}_{o(o\iota)}$ and $\mathcal{R}_{o(o\iota)}$ are **big**.
- $\mathcal{D}_{o(o(o\iota))}$ and $\mathcal{R}_{o(o(o\iota))}$ are **small**.
- $\mathcal{D}_{o(o(o(o\iota)))}$ and $\mathcal{R}_{o(o(o(o\iota)))}$ are **big**.
- etc. . .

Frame in which Injective Cantor Fails

- $\mathcal{D}_\iota$ and $\mathcal{D}_{o\iota}$ are countable.
- $\mathcal{D}_{o(o\iota)} = (\mathcal{D}_o)^{\mathcal{D}_{o\iota}}$ is uncountable (continuum).
- $\mathcal{D}_{o(o(o\iota))}$ is again *countable*.
- $\mathcal{D}_{o(o(o(o\iota)))} = (\mathcal{D}_o)^{\mathcal{D}_{o(o(o\iota))}}$ is uncountable (continuum).
- Alternates up the heirarchy.

Frame in which Injective Cantor Fails

	cardinality	standard	Π^α	Σ^α	$=^\alpha$
$\vdots$					
$\mathcal{D}_{o(o(o(o(o\iota))))}$	$\aleph_0$	no	no	no	yes
$\mathcal{D}_{o(o(o(o\iota)))}$	$2^{\aleph_0}$	yes	yes	yes	no
$\mathcal{D}_{o(o(o\iota))}$	$\aleph_0$	no	no	no	yes
$\mathcal{D}_{o(o\iota)}$	$2^{\aleph_0}$	yes	yes	yes	no
$\mathcal{D}_{o\iota}$	$\aleph_0$	no	no	no	yes
$\mathcal{D}_{\iota}$	$\aleph_0$	–	yes	yes	no

Frame in which Injective Cantor Fails

Injective Cantor Theorem is not provable even using signature

$\mathcal{S} = \{\top, \bot, \neg, \vee, \wedge\} \cup \{\Pi^{\alpha} \mid \alpha \in \mathcal{T}^{e}\} \cup \{\Sigma^{\alpha} \mid \alpha \in \mathcal{T}^{e}\} \cup \{=^{\alpha} \mid \alpha \in \mathcal{T}^{o}\}$

where

$$\mathcal{T}^{e} = \{\iota, o(o\iota), o(o(o(o\iota))), \ldots\}$$

and

$$\mathcal{T}^{o} = \{(o\alpha) \mid \alpha \in \mathcal{T}^{e}\}$$

Independence Results:

- The signature $\mathcal{S} \cup \{\Pi^{o\iota}\}$ is not conservative over $\mathcal{S}$.
- The signature $\mathcal{S} \cup \{=^{\iota}\}$ is not conservative over $\mathcal{S}$.

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The End.